P: ISSN No. 0976-8602 RNI No.UPENG/2012/42622 VOL.-8, ISSUE-1, January 2019

E: ISSN No. 2349-9443 Asian Resonance

The Global Dynamical Behavior of Food Web of Two Logistic Prey and Modified Leslie- Gower type Predator

Abstract

The dynamics of food web consisting of two logistic preys and a predator is investigated. Modified Leslie-Gower type dynamics is considered for the predator. The model is analyzed mathematically. Analysis of nonzero positive equilibrium gives conditions for persistence. Global behavior is simulated numerically for biologically feasible choice of parameters. The persistence in the form of local and global stability is investigated.

Keywords: Local Stability, Global Stability, Range of Biological Feasible Parameters.

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Introduction

Many investigations have been carried out on multi-species ecological systems comprising of food chains of variable lengths [19-21].

Review of Literature

The underlying nonlinear equations have complex dynamical behavior: Limit cycle, quasi-periodic behavior and chaos. Three species food web systems are classified into two broad categories: (i) one prey and two predators systems (ii) two prey and one predator systems. The dynamics may further include different types of interactions between the two preys or predators as the case may be. The first category of systems has been investigated by [1. 2, 4, 5, 6, 7, 9, 15-18, 19, 21]. The chaos is not frequently observed and the models reveal quasi periodic nature of the solution [4]. Due to indirect competition between two predator species, one or more species may undergo extinction. The study of coexistence becomes more important in such food webs. Due to availability of alternate prey, the chances of coexistence are enhanced. The persistence of the species has been investigated [4]. Rich dynamical behavior including chaos has been observed in these models. A two preys and a predator model with modified Leslie Gower type dynamics has been considered [4]. Investigations have been carried out with a constraint on model parameters.

Aim of the Study

This paper is devoted to a food web comprising of two logistically growing preys and a predator. Modified Leslie-Gower type dynamics is considered. The simplifying assumption of [14] is relaxed and the predator takes food from both the prey species but not necessarily in the same proportion.

The Mathematical Model

Consider two prey one predator food web system. Two prey species are assumed to grow logistically. The predator dynamics is assumed to be of modified Leslie- Gower type. The Mathematical model is given by the following non-linear system of equations [3,10, 11, 20]:

$$\frac{dX_1}{dt}=r_1X_1(1-\frac{X_1}{K})-\frac{A_1X_1X_3}{1+B_1X_1+B_2X_2}$$
$$\frac{dX_2}{dt}=r_2X_2(1-\frac{X_2}{K})-\frac{A_2X_3X_2}{1+B_1X_1+B_2X_2} \quad (1)$$
$$\frac{dX_3}{dt}=r_3X_3{}^2(1-\frac{1}{S_3+S_1X_1+S_2X_2})$$

$X_i \geq 0, i=1,2$ represent the population density of two preys and $X_3 \geq 0$ is the population density of the predator. The constants K_i, r_i, A_i, B_i and S_i, are model parameters assuming only positive values. In the model, the third equation is written according to the Leslie- Gower scheme in which the conventional carrying capacity term is being replaced by the renewable resources for the predator as $S_1X_1+S_2X_2$. Due to availability of more food (in the form of second prey species) the predator is benefited. The additional constant S_3 normalizes the residual reductions in the predator population in case of severe scarcity of food. Further, the square term signifies the fact that mating frequency is proportional to the number of males as well as that of females. The similar dynamics for the predator was considered in [ ] with the simplifying assumption that the predator takes proportional food from the two prey species. Accordingly, it was assumed that $B_i\ /\ S_i = \text{constant, i} = 1,2$. No such assumption is made in the present analysis. Although the two prey species are not directly interacting with each other, but the growth of both the prey species increases due to presence of other species as the predator is taking food from the two preys. The following dimensionless variables/ and parameters are introduced:

$$t=r_1T,\ y_i=X_i/K_i,\ y_3=X_3/K_1, w_1=A_1/r_1,\ w_2=B_1K_1, w_3=B_2K_2, w_4=r_2/r_1,$$
$$w_5=A_2K_1/r_1, w_6=r_3K_1/r_1, w_7=1/S_3,\ w_8=\alpha_1w_2,\ w_9=\alpha_2w_3, \alpha_1=S_1/S_3B_1, \alpha_2=S_2/S_3B_2$$

The system (1) is transformed to the following non-dimensional form:

$$\frac{dy_1}{dt}=y_1(1-y_1-\frac{w_1y_3}{1+w_2y_1+w_3y_2})=y_1f_1(y_1,y_2,y_3)$$
$$\frac{dy_2}{dt}=y_2[(1-y_2)w_4-\frac{w_5y_3}{1+w_3y_2+w_2y_1}]=y_2f_2(y_1,y_2,y_3) \quad (2)$$
$$\frac{dy_3}{dt}=w_6y_3^2(1-\frac{w_7}{1+w_8y_1+w_9y_2})=w_6y_3^2(1-\frac{w_7}{1+\alpha_1w_2y_1+\alpha_2w_3y_2})=y_3f_3(y_1,y_2,y_3)$$

$$w_i>0, i=1,2,3,4,5,6,7;\ \ y_i \geq 0,\ i=1,2,3;\ \alpha_1 \neq \alpha_2 .$$

Mathematical Analysis

The system can be splitted into two disconnected sub webs:

$$\frac{dy_1}{dt}=y_1(1-y_1-\frac{w_1y_3}{1+w_2y_1})=y_1g_1(y_1,y_2)$$
$$\frac{dy_3}{dt}=w_6y_3^2(1-\frac{w_7}{1+\alpha_1w_2y_1})=y_2g_2(y_1,y_2) \quad (3A)$$

$$\frac{dy_2}{dt}=y_2[(1-y_2)w_4-\frac{w_5y_3}{1+w_3y_2}]$$
$$\frac{dy_3}{dt}=w_6y_3^2(1-\frac{w_7}{1+\alpha_2w_3y_2}) \quad (3B)$$

Lemma 4.1

Consider the domain $D_1 = \{(y_1, y_3): 0 < y_1 < \bar{y}_1 < 1, 0 < y_3\}$ and $D_2 = \{(y_2, y_3): 0 < y_2 < \bar{y}_2 < 1, 0 < y_3\}$, he sub system (3A) is Kolmogorov [ ] in the domain D1 and the subsystem (3B) is Kolmogorov in the domain D2 under the following conditions:

$$w_7/(1+\alpha_1 w_2 \bar{y}_1)^2 < 1 < w_7/(1+\alpha_1 w_2 \bar{y}_1) < w_7 \qquad (4A)$$

$$w_7/(1+\alpha_2 w_3 \bar{y}_2)^2 < 1 < w_7/(1+\alpha_2 w_3 \bar{y}_2) < w_7 \text{ respectively} \qquad (4B)$$

In fact, the following rather weak condition is considered throughout our subsequent discussion

$$w_7/(1+\alpha_1 w_2 + \alpha_2 w_3) < 1 \qquad (5)$$

Three non-negative equilibrium points for the Kolmogorov system (3A) are:

$$E_1 = (0,0),\ E_2 = (1,0),\ E_3 = (y_1^*, y_3^*);\ y_1^* = (w_7 - 1)/\alpha_1 w_2;\ y_3^* = (1 - y_1^*)(1 + w_2 y_1^*)/w_1$$

The equilibrium points E_1 and E_2 in the $y_1 - y_3$ plane always exist and the linearized systems about E_1 and E_2 have a zero eigenvalue. These are non-hyperbolic saddle points.

For the local stability of system (3A) about E_3, the eigenvalues of the corresponding variational matrix should be negative. This gives the stability condition as

$$(w_2 - 1)/2w_2 < y_1^*$$

$$\text{or } w_7 > (1 - \alpha_1/2) + \alpha_1 w_2/2 \qquad (6)$$

Further, the system will admit a limit cycle in the domain under condition

$$(w_2 - 1)/2w_2 > y_1^*$$

$$\text{or } w_7 < (1 - \alpha_1/2) + \alpha_1 w_2/2 \qquad (7).$$

Similarly for the sub-system (3B) the equilibrium points in the $y_2 - y_3$ plane $(0,0)$ and $(1,0)$ are again non-hyperbolic saddle points. However the positive nonzero equilibrium point $E_4 = (\tilde{y}_2, \tilde{y}_3)$ is given by

$$\tilde{y}_2 = (w_7 - 1)/\alpha_2 w_3,$$

$$\tilde{y}_3 = \frac{w_4}{w_5}(1 - \tilde{y}_2)(1 + w_3 \tilde{y}_2)$$

It is observed that E_4 is locally asymptotically stable for the condition

$$(w_3 - 1)/2w_3 < \tilde{y}_2$$

$$\text{or } w_7 > (1 - \alpha_1/2) + \alpha_1 w_3/2 \qquad (8)$$

The sub system (3B) admits limit cycle whenever

$$(w_3 - 1)/2w_3 > \tilde{y}_2$$

$$\text{or } w_7 < (1 - \alpha_1/2) + \alpha_1 w_3/2 \qquad (9)$$

The complete system (2) admits following equilibrium points:

The equilibrium point $(0,0,0)$ is unstable node.

The axial equilibrium points $(1,0,0)$ and $(0,1,0)$ are found to be non-hyperbolic saddle points.

The stability of planar equilibrium points $E_5 = (y_1^*, 0, y_3^*)$ and $E_6 = (0, \tilde{y}_2, \tilde{y}_3)$ is the same as that of points E_3 and E_4 respectively, for the perturbations given in the respective planes.

By the Routh-Hurwitz criterion, the local stability condition for E_5 and E_6 are obtained as

$$\frac{(w_2 - 1)}{2w_2} < y_1^* < \frac{w_5 - w_1 w_4}{w_5} = \beta \qquad (10)$$

$$\text{and } \frac{(w_3 - 1)}{2w_3} < \tilde{y}_2 < \frac{w_1 w_4 - w_5}{w_5} \text{ respectively.} \qquad (11)$$

Further from (10) and (11), it is clear that only one of E_5 and E_6 will be stable. It is now concluded that if both the subsystems have stable positive equilibrium points then the 3D system may admit local stability of only one of the two planar equilibrium points and thus one of the two prey species will face extinction.

The existence of positive equilibrium point is established in the following theorem:

Theorem 4.1

The system (2) has positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ under (5) provided one of the following is satisfied:

$$w_3(w_1 w_4 - w_5) < \frac{w_1 w_4 \varepsilon}{\alpha_2};$$

$$\varepsilon = w_7 - 1 \qquad (12)$$

$$w_2(w_5 - w_1 w_4) < \frac{\varepsilon w_5}{\alpha_1} \qquad (13)$$

Proof

For nonzero equilibrium point, equating the three equations to zero and solving them we get,

$$\hat{y}_1 = 1 - \delta w_1 w_4 / \Delta \; ; \; \hat{y}_2 = 1 - \delta w_5 / \Delta \; ; \; \hat{y}_3 = \frac{1}{w_1}(1-\hat{y}_1)(1 + w_2\,\hat{y}_1 + w_3\,\hat{y}_2)$$

$$\delta = w_2\alpha_1 + w_3\alpha_2 - \varepsilon > 0; \;\; \Delta = \alpha_1 w_1 w_2 w_4 + \alpha_2 w_5 w_3 > 0; \varepsilon = w_7 - 1 > 0$$

$$\text{or } \hat{y}_1 = \frac{w_1w_4\varepsilon + w_3\alpha_2 w_5 - w_4w_1w_3\alpha_2}{\Delta}, \; \hat{y}_2 = \frac{w_5\varepsilon - w_2\alpha_1 w_5 + w_4 w_1 w_2 \alpha_1}{\Delta},$$

$$\hat{y}_3 = \frac{\delta w_4(w_1w_2w_4(\varepsilon + \alpha_1 + w_3(\alpha_1 - \alpha_2)) + w_3w_5(\varepsilon + \alpha_2 + w_2(\alpha_2 - \alpha_1)))}{\Delta^2}. \quad (14)$$

Since $0 < \hat{y}_1 < 1, \;\; 0 < \hat{y}_2 < 1$, therefore, the system will have a positive equilibrium point provided

$$\delta w_1 w_4 / \Delta < 1 \quad \text{and} \quad \delta w_5 / \Delta < 1$$

Thus the system will have a positive solution as (14) under conditions

$$w_3(w_1w_4 - w_5) < \frac{w_1w_4\varepsilon}{\alpha_2} \quad (15)$$

$$w_2(w_5 - w_1w_4) < \frac{\varepsilon w_5}{\alpha_1} \quad (16)$$

Now the two cases arise:

Case 1

When $w_5 - w_1 w_4 > 0$

It is observed that (15) will be trivially satisfied while (16) will be satisfied provided:

$$w_2(w_5 - w_1w_4) < \frac{\varepsilon w_5}{\alpha_1}$$

Thus the system has a positive solution under condition (16).

Case 2: when $w_1 w_4 - w_5 > 0$

It is observed that (16) will be trivially satisfied while (15) will be satisfied provided:

$$w_3(w_1w_4 - w_5) < \frac{w_1w_4\varepsilon}{\alpha_2}$$

Thus the system has a positive nonzero solution under condition (15) in this case.

This proves the theorem.

The following theorem gives the conditions for the local stability of the nonzero positive equilibrium point.

Theorem

The positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is locally asymptotically stable provided the following are satisfied simultaneously:

$$2(w_1w_4\varepsilon + w_5w_3\alpha_2) + \frac{\Delta}{w_2} > \Delta + 2\alpha_2 w_3 w_1 w_4 \quad (17a)$$

$$w_3(w_5\varepsilon + \alpha_1 w_1 w_4 w_2) + \Delta > \frac{w_3w_5}{w_1w_4}\Delta + \alpha_1 w_2 w_5 w_3 \quad (17b)$$

$$\varepsilon = w_7 - 1 > 0; \quad \Delta = \alpha_1 w_1 w_2 w_4 + \alpha_2 w_3 w_5$$

Proof. Assume $y_1 = \hat{y}_1 + u, y_2 = \hat{y}_2 + v, y_3 = \hat{y}_3 + w$, where u, v and w small perturbations are about $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$. The variational matrix about $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is given by

$$J = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -\hat{y}_1 + \dfrac{w_1w_2\hat{y}_1\hat{y}_3}{(1+w_2\hat{y}_1+w_3\hat{y}_2)^2} & \dfrac{w_1w_3\hat{y}_1\hat{y}_3}{(1+w_2\hat{y}_1+w_3\hat{y}_2)^2} & -\dfrac{w_1\hat{y}_1}{(1+w_2\hat{y}_1+w_3\hat{y}_2)} \\ \dfrac{w_2w_5\hat{y}_2\hat{y}_3}{(1+w_2\hat{y}_1+w_3\hat{y}_2)^2} & \hat{y}_2(-w_4 + \dfrac{w_5w_3\hat{y}_3}{(1+w_2\hat{y}_1+w_3\hat{y}_2)^2}) & -\dfrac{w_5\hat{y}_2}{(1+w_2\hat{y}_1+w_3\hat{y}_2)} \\ \dfrac{\alpha_1w_7w_2w_6\hat{y}_3\hat{y}_3}{(1+\alpha_1w_2\hat{y}_1+\alpha_2w_3\hat{y}_2)^2} & \dfrac{\alpha_2w_7w_3w_6\hat{y}_3\hat{y}_3}{(1+\alpha_1w_2\hat{y}_1+\alpha_2w_3\hat{y}_2)^2} & 0 \end{bmatrix}$$

The characteristic equation of the above variational matrix about $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is

$$\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2 = 0;$$

$$a_0 = -(a_{11} + a_{22}); \qquad a_1 = a_{11}a_{22} + a_{12}a_{21} - a_{32}a_{23} - a_{13}a_{31};$$

$$a_2 = a_{11}a_{23}a_{32} + a_{13}a_{31}a_{22} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

P: ISSN No. 0976-8602 RNI No.UPENG/2012/42622 VOL.-8, ISSUE-1, January 2019
E: ISSN No. 2349-9443 Asian Resonance

It may be observed that a1 and a2 are positive. Using Routh-Hurwitz criterion on the variational matrix *J* gives the stability conditions of the equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ as (17a) and 17b

This completes the proof of the theorem 4.2. □

The following theorem gives the conditions for the global stability of positive nonzero equilibrium point.

Theorem 4.3 The positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is globally asymptotically stable provided the following are satisfied:

$$A = (1 + w_2\hat{y}_1 + w_3\hat{y}_2 - w_2) > 0; \; B = (1 + w_2\hat{y}_1 + w_3\hat{y}_2 - w_3) > 0.$$

$$w_3^2 m^2 + w_2^2 w_4^2 < 4mw_4 AB\,; \; m = \frac{\alpha_1 w_2 w_5}{\alpha_2 w_1 w_3} \qquad (18)$$

Proof: Consider the small perturbations u, v, w about the positive unique equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ such that $y_1 = \hat{y}_1 + u, y_2 = \hat{y}_2 + v, y_3 = \hat{y}_3 + w$. Consider positive definite function for arbitrarily chosen nonzero positive constants $\mathrm{D}_1, \mathrm{D}_2 \text{and } \mathrm{D}_3$:

$$V(t) = D_1(u - \hat{y}_1 \log(1 + \frac{u}{\hat{y}_1})) + D_2(v - \hat{y}_2 \log(1 + \frac{v}{\hat{y}_2})) + D_3(w - \hat{y}_3 \log(1 + \frac{w}{\hat{y}_3}))$$

We have,

$$\frac{dV}{dt} = D_1 u[(1 - \hat{y}_1 - u) - \frac{w_1(\hat{y}_3 + w)}{1 + w_2 y_1 + w_3 y_2}] + D_2 v[(1 - \hat{y}_2 - v)w_4 - \frac{w_5(\hat{y}_3 + w)}{1 + w_2 y_1 + w_3 y_2}]$$

$$+ D_3 w y_3 w_6 [1 - \frac{w_7}{1 + \alpha_1 w_2 y_1 + \alpha_2 w_3 y_2)}]$$

$$\frac{dV}{dt} = -[A'^2 u^2 + B'^2 v^2 - C'uv] + uw[\frac{w_6 y_3 \alpha_1 w_2 D_3}{l_2} - \frac{w_1 D_1}{l_1}] + vw[\frac{w_6 y_3 \alpha_2 w_3 D_3}{l_2} - \frac{w_5 D_2}{l_1}]$$

$$\text{where} \quad A'^2 = \frac{AD_1}{l_1}; \quad B'^2 = \frac{BD_2 w_4}{l_1}; \quad C' = [\frac{w_3 D_1}{l_1} + \frac{w_2 D_2 w_4}{l_1}];$$

Substituting $(w_6 y_3 \alpha_1 w_2 D_3) l_1 = w_1 D_1 l_2$ and $(w_6 y_3 \alpha_2 w_3 D_3) l_1 = w_5 D_2 l_2$ and selecting the arbitrary constants as

$D_1 = mD_2; \quad m = (\alpha_1 w_2 w_5)/\alpha_2 w_1 w_3$, then

$$\frac{dV}{dt} = -[A'^2 u^2 + B'^2 v^2 - C'uv]$$

or

$$\frac{dV}{dt} = -A'^2 (u - \frac{C'}{2A'^2} v)^2 - (B'^2 - \frac{C'^2}{4A'^2}) v^2.$$

For $A > 0$ and $B > 0$, the expression for $\frac{dV}{dt}$ is negative definite provided

$$4A'^2 B'^2 > C'^2.$$

or

$$w_3^2 m^2 + w_2^2 w_4^2 < 2mw_4(2AB - w_2 w_3)$$

or $\quad w_3^2 m^2 + w_2^2 w_4^2 < 4mw_4 AB$

Therefore, the function V is a Liapunov function

Thus, the positive nonzero equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is globally asymptotically stable under the conditions (18).

Numerical Simulation

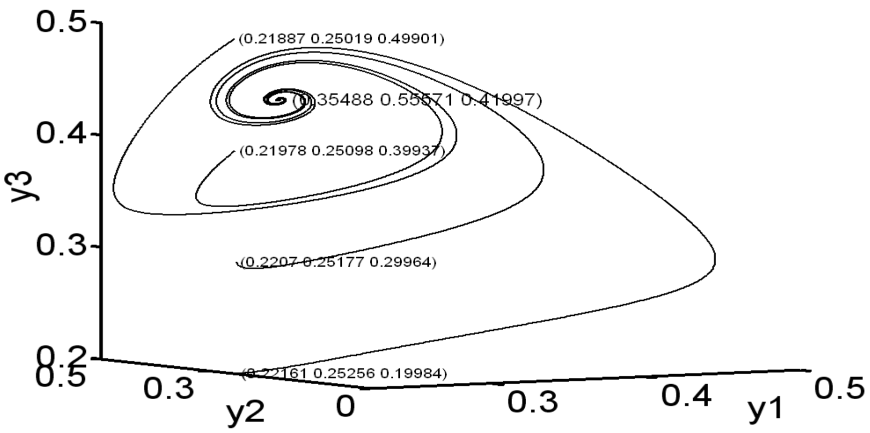
For global dynamic behavior, numerical simulations of the underlying non-linear system are carried out. The numerical values for various parameters are selected according to the mathematical restrictions (4) obtained from the Kolmogorov analysis. In all cases the weak condition (6) is satisfied. These ensure that the parameters take biologically relevant values only. As the solution of the system is bounded, the long time behavior of the solution is obtained as limit point attractor.

Numerical results for global analysis with respect to w_7 is shown for the following data:

$w_1 = 3.3, w_2 = 1.2, w_3 = 1.3, w_4 = 1.1, w_5 = 2.5, w_6 = 1.0, w_7 = 1.40, \alpha_1 = 0.9, \alpha_2 = 0.3$

Fig. Phase plot

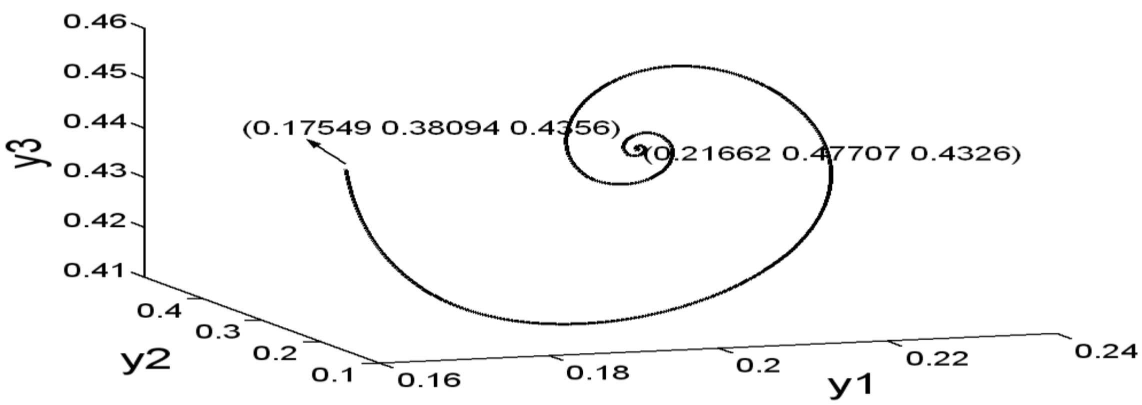
(global stable)



In Fig, the trajectories are drawn for the following data:

w_1=3.3, w_2=1.2, w_3=1.3, w_4=1.1, w_5= 2.5, w_6 =1.0, w_7 = 1.7, α_1 =1.5, α_2 =0.5;

Fig. Phase Plot (Locally Stable)



The system (2) admits a nonzero positive equilibrium point, which is stable.

Conclusion

This paper deals with the dynamics of food web consisting of two logistic preys and a predator. Modified Leslie- Gower type dynamics is considered for the predator. The global stability of the food web is examined. Numerical integration of the food-web non-linear system is carried out under the Kolmogorov biologically feasible conditions. The stability of food web and sub webs is discussed analytically.

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